n-dim 空间以 $m_1 + m_2$ 分解

$\downarrow$

dim. of the surface

$$\{x^1 \cdots x^{m_1},\ \xi^1 \cdots \xi^{m_2}\}$$

$\alpha, \beta, \gamma \cdots = 1 \cdots m_1$ $\qquad$ $i, j, k \cdots = \underset{\xi^1}{m_1+1} \cdots \underset{\xi^{m_2}}{n}$

$dx^1 \cdots$ Schmidt 正交化 $\Rightarrow n^1 \cdots n^{m_1}$

且 $n^{\alpha a} n^{\alpha}{}_a = \epsilon^{\alpha}$

$$h^a{}_b = \delta^a{}_b - \sum_\alpha \epsilon^\alpha n^{\alpha a} n^\alpha{}_b$$

$$\Rightarrow h_{ab} = (\delta^c{}_a - \epsilon^\alpha n^{\alpha c} n^\alpha{}_a)\, g_{cd}\, (\delta^d{}_b - \epsilon^\beta n^{\beta d} n^\beta{}_b)$$

$$= g_{ab} - 2\epsilon^\alpha n^\alpha{}_a n^\alpha{}_b + \epsilon^\alpha \epsilon^\beta \underbrace{n^{\alpha c} n^\beta{}_c}_{\epsilon^\alpha \delta_{\alpha\beta}} n^\alpha{}_a n^\beta{}_b$$

$$= g_{ab} - \sum_\alpha \epsilon^\alpha n^\alpha{}_a n^\alpha{}_b \qquad + \epsilon^\alpha n^\alpha{}_a n^\alpha{}_b$$

$$h^{ab} = g^{ab} - \sum_\alpha \epsilon^\alpha n^{\alpha a} n^{\alpha b}$$

有 $h^{ac} h_{cb} = (g^{ac} - \epsilon^\alpha n^{\alpha a} n^{\alpha c})(g_{cb} - \epsilon^\beta n^\beta{}_c n^\beta{}_b)$

$$= \delta^a{}_b - 2\epsilon^\alpha n^{\alpha a} n^\alpha{}_b + \epsilon^\alpha \epsilon^\beta n^{\alpha a} \underbrace{n^{\alpha c} n^\beta{}_c}_{\epsilon^\alpha \delta_{\alpha\beta}} n^\beta{}_c$$

$$= h^a{}_b$$

$$\left(\frac{\partial}{\partial x^\alpha}\right)^a = \sum_\beta N^{\alpha\beta} n^{\beta a} + M^{\alpha i} \left(\frac{\partial}{\partial \xi^i}\right)^a$$

$$g_{\alpha\beta} = g_{ab} N^{\alpha\gamma} n^{\gamma a} N^{\beta\delta} n^{\delta b} + M^{\alpha i} M^{\beta j} g_{ij}$$

$\downarrow$

$$N^{\alpha\gamma} N^{\beta\delta} \epsilon^\gamma \delta_{\gamma\delta} = \sum_\gamma \epsilon^\gamma N^{\alpha\gamma} N^{\beta\gamma}$$

$$\Rightarrow g_{\alpha\beta} = \sum_\gamma \epsilon^\gamma N^{\alpha\gamma} N^{\beta\gamma} + N^{\alpha i} N^{\beta j} g_{ij}$$

$$g_{\alpha i} = g_{ab}\left(\frac{\partial}{\partial x^{\alpha}}\right)^a\left(\frac{\partial}{\partial \xi^{i}}\right)^b = M^{\alpha j}\left(\frac{\partial}{\partial \xi^{j}}\right)^a\left(\frac{\partial}{\partial \xi^{i}}\right)^b g_{ab}$$

$$= M^{\alpha j} g_{ji}$$

$$g_{\mu\nu} = \begin{pmatrix} \epsilon^{\gamma} N^{\alpha\gamma} N^{\beta\gamma} + N^{\alpha i} N^{\beta j} g_{ij} & \{N^{\alpha j} g_{ji}\}^T \\ N^{\alpha j} g_{ji} & g_{ij} \end{pmatrix}$$

$$\leftarrow \quad -\{N^{\alpha i}\}\cdot\{N^{\alpha j} g_{ji}\}^T = -\{N^{\alpha i} N^{\beta j} g_{ij}\}$$

$$\leftarrow \quad -\{N^{\alpha i}\}\cdot\{g_{ij}\}$$

$$g_{\mu\nu} \longrightarrow \begin{pmatrix} \epsilon^{\gamma} N^{\alpha\gamma} N^{\beta\gamma} & \cdots \\ \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} & g_{ij} \end{pmatrix}$$

$$\det\{g_{\mu\nu}\} = \det\{\epsilon^{\gamma} N^{\alpha\gamma} N^{\beta\gamma}\} \ \det\{g_{ij}\}$$

且 $g_{ij} \equiv h_{ij}$

令 $|\det\{g_{\mu\nu}\}| = g$, $\quad \det\{\epsilon^{\gamma} N^{\alpha\gamma} N^{\beta\gamma}\} = \underset{\downarrow\ \pm 1}{\epsilon}\, N^2 \quad |\det\{h_{ij}\}| = h$

then we have : $g = N^2 h$ (取 $|\ |$ 就没有 $\epsilon$ 了)

$$\Rightarrow \quad \sqrt{h} = \frac{\sqrt{g}}{N}$$

induced volume form :

$$\tilde{\epsilon} = \sqrt{h}\ \ h_{a_1}{}^{b_1}(d\xi^1)_{b_1} \wedge \cdots \wedge h_{a_{m_2}}{}^{b_{m_2}}(d\xi^{m_2})_{b_{m_2}}$$

$$\downarrow$$

$$(d\xi^1)_{a_1} - \sum_{\alpha} \epsilon^{\alpha} n^{\alpha}{}_{a_1} n^{\alpha b_1}(d\xi^1)_{b_1}$$

$$= \frac{\sqrt{g}}{N}\Big((d\xi^1)_{a_1} \wedge \cdots (d\xi^{m_2})_{a_n} - \epsilon^{\alpha} n^{\alpha b_1}(d\xi^1)_{b_1} \ n_{a_1} \wedge (d\xi^2)_{a_2} \wedge \cdots (d\xi^{m_2})_{a_{m_2}} - \cdots\Big)$$

而 $\epsilon = \sqrt{g}\; dx^1\wedge\cdots dx^{m_1}\wedge d\xi^1\wedge\cdots d\xi^{m_2}$

$$dx^1\wedge\cdots dx^{m_1} = \frac{1}{N}\, n^1\wedge\cdots\wedge n^{m_1}$$

所以 $n^1\wedge\cdots\wedge n^{m_1}\wedge\tilde{\epsilon} = \epsilon$

当然，我们还要证明: $dx^1\wedge\cdots\wedge dx^{m_1} = \frac{1}{N}\, n^1\wedge\cdots\wedge n^{m_1}$

where $dx^\alpha \underset{\text{Schmidt 正交化}}{\longmapsto} n^\alpha$   assume : $n^\alpha = \sum_\beta \lambda^{\alpha\beta}\, dx^\beta$

and $\epsilon N^2 = \det\{\epsilon^\gamma N^{\alpha\gamma} N^{\beta\gamma}\}$

$$\Downarrow$$

$$g_{ab}\, n^{\alpha a} n^{\beta b} = g_{ab}\, \lambda^{\alpha\gamma}(dx^\gamma)^a \lambda^{\beta\delta}(dx^\delta)^b$$
$$= g^{\gamma\delta}\lambda^{\alpha\gamma}\lambda^{\beta\delta}$$
$$= (\epsilon^\alpha)\,\delta^{\alpha\beta}$$
$$\Downarrow$$
$$\lambda\cdot\{g^{\alpha\beta}\}\cdot\lambda^T = \begin{pmatrix}\epsilon^1 & & \\ & \ddots & \\ & & \epsilon^{m_1}\end{pmatrix}$$

$$N^{\alpha\beta} = \epsilon^\beta\, n^\beta{}_a \left(\frac{\partial}{\partial x^\alpha}\right)^a$$
$$= \epsilon^\beta\, \lambda^{\beta\gamma}(dx^\gamma)_a \left(\frac{\partial}{\partial x^\alpha}\right)^a$$
$$= \epsilon^\beta\, \lambda^{\beta\alpha}$$
$$\Downarrow$$
$$\{N^{\alpha\beta}\}^T = \mathrm{diag}\{\epsilon^1\cdots\epsilon^{m_1}\}\cdot\lambda$$

$$\epsilon^\gamma N^{\alpha\gamma} N^{\beta\gamma} = N\cdot\mathrm{diag}\{\epsilon^1\cdots\}\cdot N^T$$
$$= \lambda^T\cdot\mathrm{diag}\{\epsilon^1\cdots\}^3\cdot\lambda$$

$\mathrm{diag}\{\epsilon^1\cdots\}^2 = I$

$$\equiv \lambda^T\cdot\mathrm{diag}\{\epsilon^1\cdots\}\cdot\lambda$$

$\Rightarrow\ \epsilon N^2 = (\det\lambda)^2 \prod_\alpha \epsilon^\alpha$   Schmidt 正交化保证 $\det\lambda > 0$

所以 $N = \det\lambda$

且有 $n^1\wedge\cdots\wedge n^{m_1} = \lambda^{1\alpha_1}(dx^{\alpha_1})\wedge\cdots\wedge\lambda^{m_1\alpha_{m_1}}\, dx^{\alpha_{m_1}}$

$$= \epsilon^{\alpha_1\cdots\alpha_{m_1}}\lambda^{1\alpha_1}\cdots\lambda^{m_1\alpha_{m_1}}\; dx^1\wedge\cdots\wedge dx^{m_1}$$

$$= \underbrace{\det\lambda}_{N}\; dx^1\wedge\cdots dx^{m_1}$$   得证